

What can we learn from three-pion interferometry?

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Abstract

We address the question which additional information on the source shape and dynamics can be extracted from three-particle Bose-Einstein correlations. For chaotic sources the true three-particle correlation term is shown to be sensitive to the momentum dependence of the saddle point of the source and to its asymmetries around that point. For partially coherent sources the three-pion correlator allows to measure the degree of coherence without contamination from resonance decays. We derive the most general Gaussian parametrization of the two- and three-particle correlator for this case and discuss the space-time interpretation of the corresponding parameters.

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I. INTRODUCTION

Two-particle Bose-Einstein interferometry (also known as Hanbury Brown-Twiss intensity interferometry) as a method for obtaining information on the space-time geometry and dynamics of relativistic heavy ion collisions has recently received intensive theoretical and experimental attention. Detailed theoretical investigations (for a recent review see Ref. [1]) have shown that high-quality two-particle correlation data can reveal not only the geometric

extension of the particle-emitting source but also its dynamical state at particle freeze-out. This information is encoded in the second central space-time moments of the “emission function” $S(x, K)$, i.e. of the Wigner phase-space density of the source. For chaotic sources, certain linear combinations of these moments can be extracted from the two-particle correlation function $C_2(\mathbf{q}, \mathbf{K})$ by fitting it to a Gaussian in the relative momentum q of the pair [2–4]. These second space-time moments give the size of the regions of homogeneity [5,3] which effectively contribute to the emission of particle pairs with a given pair momentum K ; collective dynamics of the source results in a characteristic K -dependence of these homogeneity regions [6,7,4].

More detailed information on the space-time structure of the source may be hidden [8] in possible non-Gaussian features of the correlation function $C_2(\mathbf{q}, \mathbf{K})$ even if they are hard to extract; due to the symmetry under $q \rightarrow -q$, however, only even space-time moments of the source are accessible via two-particle correlations. In this paper we will extend previous studies of multi-particle correlations [9–13] and show that three-pion correlations provide in principle additional information on the space-time characteristics of the source which cannot be obtained from two-particle interferometry. We show in particular in Sec. II that for completely chaotic sources the true three-pion correlations are determined by the phase of the two-particle exchange amplitude [13,14] which drops out from the two-particle cross section. This phase is shown to be sensitive to the rate at which the saddle point $\bar{x}(K)$ of the source, from which most pairs with momentum K are emitted, moves as K changes, and to the asymmetries of the emission function around this saddle point via its *third* central space-time moments. Unfortunately, this phase turns out to be generically small, and its sensitivity to these asymmetries is very weak, making them extremely hard to measure.

In the absence of such a non-trivial phase, three-particle correlations can still be used to test the chaoticity of the emitting source. To this end we derive the expressions for two- and three-particle correlations for chaotic and partially coherent sources and establish their respective relationships. Our treatment differs from previous studies of multi-particle Bose-Einstein correlations in that we consistently express the correlation functions through

the source Wigner density, even for partially coherent sources. This enables us to relate the shape of the correlators as functions of the various relative momenta to certain space-time features of the source. To the best of our knowledge the corresponding relations for partially coherent sources (Eqs. (34) - (36)) are new.

II. CHAOTIC SOURCES

For a chaotic source, the two-pion correlation function $C_2(p_i, p_j)$ can be expressed as [1,13]

$$\begin{aligned} C_2(p_i, p_j) &= \frac{P_2(p_i, p_j)}{P_1(p_i) P_1(p_j)} \\ &= 1 + \frac{|\int d^4x S(x, K_{ij}) e^{iq_{ij} \cdot x}|^2}{\int d^4x S(x, p_i) \int d^4y S(y, p_j)} = 1 + \frac{|\rho_{ij}|^2}{\rho_{ii} \rho_{jj}}. \end{aligned} \quad (1)$$

Here $P_2(p_i, p_j)$ is the two-pion inclusive cross section, and $P_1(p_i)$ is the single-particle inclusive spectrum. $S(x, p)$ is the single-particle Wigner density of the source, i.e. the quantum mechanical analogue of its phase-space distribution. The average and relative 4-momenta $K_{ij} = (p_i + p_j)/2$ and $q_{ij} = p_i - p_j$ satisfy the constraint $q_{ij} \cdot K_{ij} = 0$ which results from the on-shell nature of the observed momenta p_i . The two-particle exchange amplitude ρ_{ij} is defined as [13,15]

$$\begin{aligned} \rho_{ij} &= \rho(q_{ij}, K_{ij}) = \sqrt{E_i E_j} \langle \hat{a}^\dagger(p_i) \hat{a}(p_j) \rangle \\ &= \int d^4x S(x, K_{ij}) e^{iq_{ij} \cdot x} \equiv f_{ij} e^{i\phi_{ij}}. \end{aligned} \quad (2)$$

From (2) it follows that $\rho_{ij} = \rho_{ji}^*$ and thus $f_{ij} = f_{ji}$ and $\phi_{ij} = -\phi_{ji}$. Correspondingly, $\phi_{ii} = 0$, $\rho_{ii} = f_{ii}$, and f_{ij} must be an even function of q_{ij} while ϕ_{ij} is odd in q_{ij} .

The single-pion spectrum can be written as

$$P_1(p_i) = \int d^4x S(x, p_i) = f_{ii} \quad (3)$$

while the true two-pion correlation function is defined by

$$R_2(i, j) \equiv R_2(p_i, p_j) = C_2(p_i, p_j) - 1 = \frac{f_{ij}^2}{f_{ii} f_{jj}}. \quad (4)$$

Similarly, the true three-pion correlation function is given by [13,16–21]

$$\begin{aligned}
R_3(p_1, p_2, p_3) &= C_3(p_1, p_2, p_3) - R_2(1, 2) - R_2(2, 3) - R_2(3, 1) - 1 \\
&= 2 \frac{\text{Re}(\rho_{12} \rho_{23} \rho_{31})}{f_{11} f_{22} f_{33}} \\
&= 2 \frac{f_{12} f_{23} f_{31}}{f_{11} f_{22} f_{33}} \cos(\phi_{12} + \phi_{23} + \phi_{31}).
\end{aligned} \tag{5}$$

Since the real parts f_{ij} of the exchange amplitudes ρ_{ij} can be extracted from the two-pion correlator, for chaotic sources the only additional information contained in the 3-pion correlation function resides in the phase [13]

$$\Phi \equiv \phi_{12} + \phi_{23} + \phi_{31}; \tag{6}$$

it is a linear combination of the phases of the three exchange amplitudes ρ_{12} , ρ_{23} , and ρ_{31} which enter the true 3-pion correlator R_3 . This phase is odd under interchange of any two particles. It can be isolated by normalizing R_3 with respect to the true 2-pion correlator R_2 :

$$r_3(p_1, p_2, p_3) = \frac{R_3(p_1, p_2, p_3)}{\sqrt{R_2(1, 2)R_2(2, 3)R_2(3, 1)}} = 2 \cos \Phi. \tag{7}$$

In order to understand which space-time features of the source affect the phase Φ (and thus the normalized true 3-pion correlation function r_3) we expand the exchange amplitude ρ_{ij} for small values of $q_{ij} = p_i - p_j$ [2,3]. We define the average of an arbitrary space-time function $f(x)$ with the source distribution $S(x, K_{ij})$ as

$$\langle f(x) \rangle_{ij} = \frac{\int d^4x f(x) S(x, K_{ij})}{\int d^4x S(x, K_{ij})}. \tag{8}$$

This average is a function of the pair momentum K_{ij} . Using (2) we thus get

$$\rho_{ij} = P_1(K_{ij}) \left[1 + i \langle q_{ij} \cdot x \rangle_{ij} - \frac{1}{2} \langle (q_{ij} \cdot x)^2 \rangle_{ij} - \frac{i}{6} \langle (q_{ij} \cdot x)^3 \rangle_{ij} + O(q_{ij}^4) \right]. \tag{9}$$

Separating real and imaginary parts we find, after a little algebra,

$$f_{ij} = P_1(K_{ij}) \left[1 - \frac{1}{2} \langle (q_{ij} \cdot \tilde{x}_{ij})^2 \rangle_{ij} + O(q_{ij}^4) \right] \tag{10}$$

and

$$\phi_{ij} = q_{ij} \cdot \langle x \rangle_{ij} - \frac{1}{6} \langle (q_{ij} \cdot \tilde{x}_{ij})^3 \rangle_{ij} + O(q_{ij}^5), \quad (11)$$

where

$$\tilde{x}_{ij} = x - \langle x \rangle_{ij} = x - \bar{x}(K_{ij}) \quad (12)$$

is the distance to the “saddle point” of the source, i.e. to the point of maximum emission for pions with momentum K_{ij} . According to Eqs. (10) and (4), the two-pion correlator is sensitive to the second central (i.e. saddle-point corrected) space-time moments of the emission function $S(x, K_{ij})$ [2,3], with higher order corrections from all even central space-time moments. The phase Φ , on the other hand, contains information on the odd space-time moments. Expanding $S(x, K_{ij})$ around the average momentum K of the pion triplet,

$$K = \frac{p_1 + p_2 + p_3}{3} = \frac{K_{12} + K_{23} + K_{31}}{3}, \quad (13)$$

$$K_{ij} = K + \frac{1}{6} (q_{ik} + q_{jk}), \quad i \neq j \neq k, \quad (14)$$

and using $q_{12} + q_{23} + q_{31} = 0$, we find from Eqs. (6) and (11)

$$\begin{aligned} \Phi = & \frac{1}{2} q_{12}^\mu q_{23}^\nu \left[\frac{\partial \langle x_\mu \rangle}{\partial K^\nu} - \frac{\partial \langle x_\nu \rangle}{\partial K^\mu} \right] \\ & - \frac{1}{24} \left[q_{12}^\mu q_{12}^\nu q_{23}^\lambda + q_{23}^\mu q_{23}^\nu q_{12}^\lambda \right] \left[\frac{\partial^2 \langle x_\mu \rangle}{\partial K^\nu \partial K^\lambda} + \frac{\partial^2 \langle x_\nu \rangle}{\partial K^\lambda \partial K^\mu} + \frac{\partial^2 \langle x_\lambda \rangle}{\partial K^\mu \partial K^\nu} \right] \\ & - \frac{1}{2} q_{12}^\mu q_{23}^\nu (q_{12} + q_{23})^\lambda \langle \tilde{x}_\mu \tilde{x}_\nu \tilde{x}_\lambda \rangle + O(q^4). \end{aligned} \quad (15)$$

Here the average without subscripts

$$\langle f(x) \rangle = \frac{\int d^4x f(x) S(x, K)}{\int d^4x S(x, K)} \quad (16)$$

denotes the space-time average with the emission function evaluated at the mean momentum K of the pion triplet, and

$$\tilde{x} = x - \langle x \rangle = x - \bar{x}(K). \quad (17)$$

Eq. (15) is the main new result of this Section. One easily checks that it has the correct symmetries under particle exchange. It should be noted that, due to the on-shell constraint $q_{ij} \cdot K_{ij} = 0$, only three of the four components q_{ij}^μ are independent. The resulting relation

$$(q^0)_{ij} = \mathbf{q}_{ij} \cdot \boldsymbol{\beta}_{ij}, \quad \text{with} \quad \boldsymbol{\beta}_{ij} = \mathbf{K}_{ij}/(K^0)_{ij}, \quad (18)$$

can be used to eliminate the redundant q -components in Eq. (15), thereby mixing spatial and temporal components of the corresponding coefficients. This is a well-known problem also for the two-pion correlator (see, e.g., [1]) which prohibits a clean model-independent separation of the spatial and temporal widths of the source.

Eq. (15) features two types of contributions to the phase Φ : The formally leading contribution enters at second order in the relative momenta q_{ij} and is proportional to the rate $\partial \bar{x}_\mu(K)/\partial K^\nu$ with which the saddle point of the emission function changes as a function of the pion momentum K . This term will in general be non-zero even for emission functions with a purely Gaussian x -dependence. It gives rise to a leading q^4 -dependence of the normalized true three-particle correlator $r_3 = 2 \cos \Phi$. At order q^3 the phase Φ receives additional contributions from the second K -derivatives of the saddle point as well as from the third central space-time moments $\langle \tilde{x}_\mu \tilde{x}_\nu \tilde{x}_\lambda \rangle$ of the source. The latter are the leading contributions from a possible asymmetry of the emission function $S(x, K)$ around its saddle point $\bar{x}(K)$; they vanish for purely Gaussian emission functions. We see that they enter the normalized three-particle correlator r_3 at order q^5 in a mixture with the K -dependence of the saddle point. This renders their isolation essentially impossible.

In contrast to the widths of the emission function, which affect the two-pion correlator at *second* order in the relative momentum, the additional structural information which can (in principle) be extracted from the (normalized) three-pion correlator is seen to enter at most at *fourth* order in q . Their measurement is thus very sensitive to an accurate removal of all leading q^2 -dependences by proper normalization to the two-particle correlators. To achieve this looks like an extremely difficult experimental task. We are therefore somewhat pessimistic about the short-term prospects of extracting additional structural information about the source from three-pion correlations.

If the phase Φ and the information it contains about the source are inaccessible, what else can three-pion correlations be used for experimentally? The answer is that one can test

the assumption that the source is chaotic. This has been pointed out previously in Refs. [17,19] where specific simple parametrizations for the two- and three-particle correlators (as well as for higher order correlations) were assumed and the relationship between the various parameters was studied. We will here derive more general expressions which, in principle, permit such a test without making any simplifying assumptions about the shape of the source.

Before proceeding to the discussion of Bose-Einstein correlations from partially coherent sources, we would like to close this Section with a few short remarks on the effects from resonance decays. It is well known [22,23] that partial coherence in the source leads to incomplete correlations in the two-particle sector, in the sense that $R_2(q, K)$ at vanishing relative momentum $q = 0$ does not approach the ideal value $R_2(0, K) = 2$ for chaotic sources. In actual experiments there are, however, other possible reasons for apparently incomplete two-particle correlations. Most importantly, pions from the decay of long-lived resonances contribute to the correlator only at very small values of q and thus (due to limited 2-track resolution) may escape detection in the correlation signal while fully contributing to the single-particle spectrum, thereby reducing the apparent correlation strength even for a completely chaotic source [24–27]. In a Gaussian parametrization of the exchange amplitude this can be implemented by writing instead of Eq. (10) for $q_{ij} \neq 0$

$$f_{ij} = \lambda^{1/2}(K_{ij}) P_1(K_{ij}) \exp \left[-\frac{1}{2} q_{ij}^\mu q_{ij}^\nu R_{\mu\nu}(K_{ij}) \right], \quad (19)$$

where, up to second order in q , $R^{\mu\nu}(K_{ij}) = \langle \tilde{x}_{ij}^\mu \tilde{x}_{ij}^\nu \rangle_{ij}$, with the source average on the r.h.s. being taken only over the “core” of pions from direct emission and from the decays of short-lived resonances [1,27,28]. The two-particle correlator then becomes

$$R_2(i, j) = \lambda(K_{ij}) \frac{P_1^2(K_{ij})}{P_1(p_i) P_1(p_j)} \exp \left[-q_{ij}^\mu q_{ij}^\nu R_{\mu\nu}(K_{ij}) \right], \quad (20)$$

and for vanishing relative momenta q the three-particle correlation function assumes the value

$$C_3(p_1=p_2=p_3=K) = 1 + 3 \lambda(K) + 2 \lambda^{3/2}(K). \quad (21)$$

Note, however, that the expression (7) for the normalized true three-pion correlation function is not affected by resonance decay contributions and remains unchanged. This will no longer be true for partially coherent sources.

III. PARTIALLY COHERENT SOURCES

Expressions for the n -particle inclusive spectra from partially coherent sources have been previously derived, with differing methods, in Refs. [13,16–20]. In the covariant current formalism of Refs. [23,15] one decomposes the classical source current which creates the free pions in the final state into a coherent and a chaotic term:

$$J(x) = J_{\text{coh}}(x) + J_{\text{cha}}(x). \quad (22)$$

Following the treatment of Ref. [15] this leads to the following definition of the single-particle Wigner density (“emission function”) of the source:

$$\begin{aligned} S(x, K) &= \int \frac{d^4 y}{2(2\pi)^3} e^{-iK \cdot y} \langle J^*(x + \frac{y}{2}) J(x - \frac{y}{2}) \rangle \\ &= S_{\text{coh}}(x, K) + S_{\text{cha}}(x, K), \end{aligned} \quad (23)$$

with

$$S_{\text{coh}}(x, K) = \int \frac{d^4 y}{2(2\pi)^3} e^{-iK \cdot y} J_{\text{coh}}^*(x + \frac{y}{2}) J_{\text{coh}}(x - \frac{y}{2}), \quad (24a)$$

$$S_{\text{cha}}(x, K) = \int \frac{d^4 y}{2(2\pi)^3} e^{-iK \cdot y} \langle J_{\text{cha}}^*(x + \frac{y}{2}) J_{\text{cha}}(x - \frac{y}{2}) \rangle. \quad (24b)$$

The average on the r.h.s. of the definition (24b) for the chaotic part of the emission function is defined as in Ref. [15], and we used

$$\langle J_{\text{cha}}^*(x) J_{\text{coh}}(y) \rangle = 0. \quad (25)$$

The Wigner density of the full source is thus the sum of a coherent and a chaotic contribution; no mixed terms occur because the chaotic and coherent source currents do not interfere. This allows to carry over the intuitive and very successful Wigner function language for fully chaotic sources to the case of partially or completely coherent sources.

We now write

$$\begin{aligned}\rho_{ij} &= \int d^4x S(x, K_{ij}) e^{iq_{ij} \cdot x} \\ &= \rho_{ij}^{\text{cha}} + \rho_{ij}^{\text{coh}} \equiv F_{ij} e^{i\Phi_{ij}} + f_{ij} e^{i\phi_{ij}},\end{aligned}\tag{26}$$

where $K_{ij} = (p_i + p_j)/2$, $q_{ij} = p_i - p_j$, and

$$F_{ij} e^{i\Phi_{ij}} = \int d^4x S_{\text{cha}}(x, K_{ij}) e^{iq_{ij} \cdot x},\tag{27a}$$

$$f_{ij} e^{i\phi_{ij}} = \int d^4x S_{\text{coh}}(x, K_{ij}) e^{iq_{ij} \cdot x}.\tag{27b}$$

As shown in Ref. [13] this yields the two-pion correlation function in the form

$$C_2(p_i, p_j) = 1 + R_2(i, j) = 1 + \frac{F_{ij}^2 + 2f_{ij}F_{ij} \cos(\Phi_{ij} - \phi_{ij})}{(f_{ii} + F_{ii})(f_{jj} + F_{jj})},\tag{28}$$

while the three-particle correlation is given by

$$\begin{aligned}C_3(p_1, p_2, p_3) &= \frac{P_3(p_1, p_2, p_3)}{P_1(p_1) P_1(p_2) P_1(p_3)} = 1 + R_2(1, 2) + R_2(2, 3) + R_2(3, 1) \\ &+ \frac{2}{P_1(p_1) P_1(p_2) P_1(p_3)} \left(F_{12} F_{23} F_{31} \cos(\Phi_{12} + \Phi_{23} + \Phi_{31}) \right. \\ &+ f_{12} F_{23} F_{31} \cos(\phi_{12} + \Phi_{23} + \Phi_{31}) \\ &+ F_{12} f_{23} F_{31} \cos(\Phi_{12} + \phi_{23} + \Phi_{31}) \\ &\left. + F_{12} F_{23} f_{31} \cos(\Phi_{12} + \Phi_{23} + \phi_{31}) \right).\end{aligned}\tag{29}$$

Similar expressions were derived in Ref. [18]. The two- and three-particle correlations are seen to vanish for completely coherent sources ($F_{ij} \rightarrow 0 \forall i, j$). In the opposite limit ($f_{ij} \rightarrow 0 \forall i, j$) one recovers the results from Sec. II for completely chaotic sources.

The representations (26) and (27) permit us to write down for F_{ij} , f_{ij} and Φ_{ij} , ϕ_{ij} similar small- q expansions as in Eqs. (10) and (11); the corresponding averages are defined with respect to the chaotic and coherent parts, respectively, of the Wigner function (23). In the true two-pion correlation function $R_2(i, j)$ of Eq. (28), the first term thus contains information on the second central space-time moments of $S_{\text{cha}}(x, K_{ij})$ while the second term mixes the second moments of $S_{\text{cha}}(x, K_{ij})$ and $S_{\text{coh}}(x, K_{ij})$ in a rather nontrivial way. Since

the number of measurable parameters in $R_2(i, j)$ is the same as before, this implies a relative loss of information: the second space-time moments of S_{cha} and S_{coh} can neither be separated nor do they simply combine to the second central moments of the total source $S = S_{\text{cha}} + S_{\text{coh}}$.

This complication goes hand in hand with a similar one in the three-pion correlator: Defining the true three-pion correlator as before,

$$\begin{aligned}
R_3(1, 2, 3) &= C_3(p_1, p_2, p_3) - 1 - R_2(1, 2) - R_2(2, 3) - R_2(3, 1) \\
&= \frac{2}{(f_{11} + F_{11})(f_{22} + F_{22})(f_{33} + F_{33})} \\
&\quad \times \left(F_{12}F_{23}F_{31} \cos(\Phi_{12} + \Phi_{23} + \Phi_{31}) + f_{12}F_{23}F_{31} \cos(\phi_{12} + \Phi_{23} + \Phi_{31}) \right. \\
&\quad \left. + F_{12}f_{23}F_{31} \cos(\Phi_{12} + \phi_{23} + \Phi_{31}) + F_{12}F_{23}f_{31} \cos(\Phi_{12} + \Phi_{23} + \phi_{31}) \right), \quad (30)
\end{aligned}$$

one sees that, in contrast to Eq. (7) for chaotic sources, the phase factors can no longer be isolated by normalizing R_3 with a proper combination of two-particle correlators R_2 . This means that, in a small- q expansion, $R_3(1, 2, 3)$ contains leading terms of second order in q which are independent of those occurring in the two-particle correlator. On the one hand, those terms supplement the incomplete information from R_2 on the second space-time moments of the source; on the other hand, they render the measurements of source asymmetries impossible.

The full reconstruction of all the (in principle) measurable information obviously requires a measurement of $R_2(i, j)$ and $R_3(1, 2, 3)$ as a function of all nine components of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. In view of the technical complexity (both experimental and theoretical) of such a program this is not likely to happen soon. It must, however, be mentioned that simple one- or two-parameter Gaussian parametrizations as suggested in Refs. [17,19,14] are not sufficient for this purpose because they very strongly prejudice the form of the source.

To pursue this last point a little further, let us define the (momentum-dependent) chaotic fraction of the single particle spectrum

$$\epsilon(p_i) = \frac{F_{ii}}{f_{ii} + F_{ii}} = \frac{\int d^4x S_{\text{cha}}(x, p_i)}{\int d^4x S(x, p_i)} \quad (31)$$

The coherent fraction is accordingly $f_{ii}/(f_{ii} + F_{ii}) = 1 - \epsilon(p_i)$. For vanishing relative momentum $q_{ij} = 0$ ($i, j = 1, 2, 3$), we then have

$$\begin{aligned} R_2(p, p) &= \epsilon(p)(2 - \epsilon(p)) , \\ R_3(p, p, p) &= 2 \epsilon^2(p) (3 - 2\epsilon(p)) . \end{aligned} \quad (32)$$

For completely chaotic sources, $\epsilon(p) = 1$, we recover the results of Sec. II. For partially coherent sources, the normalized three pion correlator r_3 at vanishing q is given by

$$r_3(p, p, p) = \frac{R_3(p, p, p)}{(R_2(p, p))^{3/2}} = 2\sqrt{\epsilon(p)} \frac{(3 - 2\epsilon(p))}{(2 - \epsilon(p))^{3/2}} \quad (33)$$

which, in general, deviates from the chaotic limit $r_3(p, p, p) = 2$.

It would thus seem to be a simple matter to check the limits of R_2 and R_3 for vanishing relative momenta and construct the ratio (33) in order to see whether or not the source contains a coherent component. In practice, however, the $q = 0$ limit can not be measured directly, but requires an extrapolation of data at finite q to zero relative momenta. It is well known that such an extrapolation can be very sensitive to the assumed functional behavior of the correlator at small q . As we will now show our results provide a basis for a reasonable parametrization of R_2 and R_3 for small q .

To this end we start from Eqs. (28) and (30) together with the small q expansions (10), (11). Noting that R_2 must vanish for $q \rightarrow \infty$, a parametrization which is correct up to second order in q is given by

$$\begin{aligned} R_2(i, j) &\approx \epsilon^2(K_{ij}) \exp \left[-q_{ij}^\mu q_{ij}^\nu R_{\mu\nu}(K_{ij}) \right] \\ &+ 2\epsilon(K_{ij})(1 - \epsilon(K_{ij})) \exp \left[-\frac{1}{2} q_{ij}^\mu q_{ij}^\nu (R_{\mu\nu}(K_{ij}) + r_{\mu\nu}(K_{ij})) \right] \cos(q_{ij} \cdot s(K_{ij})) . \end{aligned} \quad (34)$$

It follows from Eqs. (10), (11) that here

$$R^{\mu\nu}(K_{ij}) = \langle \tilde{x}_{ij}^\mu \tilde{x}_{ij}^\nu \rangle_{ij}^{\text{cha}} , \quad (35a)$$

$$r^{\mu\nu}(K_{ij}) = \langle \tilde{x}_{ij}^\mu \tilde{x}_{ij}^\nu \rangle_{ij}^{\text{coh}} , \quad (35b)$$

$$s^\mu(K_{ij}) = \langle x^\mu \rangle_{ij}^{\text{cha}} - \langle x^\mu \rangle_{ij}^{\text{coh}} . \quad (35c)$$

Eq. (34) neglects an additional factor $P^2(K_{ij})/P(p_i)P(p_j)$ which is unity for exponential single particle spectra [3]. Eq. (34) differs from the parametrization suggested in Ref. [19] by the factor $\cos(q_{ij} \cdot s(K)) \exp\left[-\frac{1}{2}q_{ij}^\mu q_{ij}^\nu r_{\mu\nu}(K_{ij})\right]$; the parametrization of Ref. [19] is thus not general enough. (It essentially assumes that the coherent part of the source is pointlike (in space *and* time!) and localized at the saddle point of the chaotic part of the source.) Note that from Eq. (34) one must still eliminate the redundant q -component via the on-shell constraint (18).

The three-pion correlator can similarly be parametrized as

$$\begin{aligned}
R_3(p_1, p_2, p_3) = & 2\epsilon^2(K) \exp\left[-\left(q_{12}^\mu q_{12}^\nu + q_{23}^\mu q_{23}^\nu + \frac{1}{2}(q_{12}^\mu q_{23}^\nu + q_{12}^\nu q_{23}^\mu)\right) R_{\mu\nu}(K)\right] \\
& \times \left[\epsilon(K) \right. \\
& + (1 - \epsilon(K)) \cos(q_{12} \cdot s(K)) \exp\left(\frac{1}{2}q_{12}^\mu q_{12}^\nu (R_{\mu\nu}(K) - r_{\mu\nu}(K))\right) \\
& + (1 - \epsilon(K)) \cos(q_{23} \cdot s(K)) \exp\left(\frac{1}{2}q_{23}^\mu q_{23}^\nu (R_{\mu\nu}(K) - r_{\mu\nu}(K))\right) \\
& + (1 - \epsilon(K)) \cos((q_{12} + q_{23}) \cdot s(K)) \\
& \left. \times \exp\left(\frac{1}{2}(q_{12} + q_{23})^\mu (q_{12} + q_{23})^\nu (R_{\mu\nu}(K) - r_{\mu\nu}(K))\right) \right]. \quad (36)
\end{aligned}$$

This again generalizes the parametrizations given in Refs. [17,19]; according to Eqs. (10), (11), it is correct up to the second order in q if one approximates $P_1^2(K_{ij})/P_1(p_i)/P_1(p_j) \approx 1$ as well as $\epsilon(K_{ij}) \approx \epsilon(K)$. The parametrizations of Ref. [17,19] are recovered in the limit of a pointlike coherent source, $r_{\mu\nu}(K) = 0$, and assuming $s(K) = 0$. (The first of these two assumptions is explicitly stated in Ref [17].) One can easily convince oneself that at $q_{12} = 0$, for example, the term $\cos(q_{23} \cdot s(K)) \exp\left[\frac{1}{2}q_{23}^\mu q_{23}^\nu (R_{\mu\nu}(K) - r_{\mu\nu}(K))\right]$ enters $R_3(q_{23})$ with a different weight than $R_2(q_{23})$. Thus R_3 provides additional information which allows to separate $R_{\mu\nu}(K)$ from $r_{\mu\nu}(K)$ and thereby the widths of the chaotic and coherent parts of the source.

In practice, one must also take into account resonance decays. Since it follows from the discussion at the end of in Sec. II that the longlived resonances do not affect the intercept (33) of the normalized true three-pion correlator, and it was shown in Refs. [1,27,28] that

expression (35a) remains essentially valid if the chaotic part of the emission function is restricted to the “core” of direct pions and short-lived resonance decays, we expect Eqs. (34) - (36) to be practically useful even when resonance decays are included.

IV. CONCLUSIONS

We have studied the question to what extent three-pion Bose-Einstein correlations can provide independent information about the space-time structure of the emitting source which cannot be extracted from two-pion correlations. For chaotic sources we found that the three-pion correlator depends on the phase of the two-particle exchange amplitude which drops out from the two-particle cross section. This phase can be isolated by proper normalization of the true three-pion correlator with respect to the two-pion correlator. It was shown to be sensitive to the momentum dependence of the point of highest emissivity in the source and to the asymmetries of the emission function around that point. However, this sensitivity is weak (it enters only at 4th order in the relative momenta q_{ij}), and the corresponding source properties are hard to measure.

We then proceeded to study sources which are not completely chaotic but contain a coherent component. We showed that in this case the emission function can be written as a sum of two Wigner densities describing the chaotic and coherent components, respectively, and expressed the two- and three-pion correlation functions via these chaotic and coherent Wigner densities. We showed that a comparison of two- and three-pion correlators allows for a determination of the degree of coherence in the source, without contaminations from resonance decays. To this end one must study the respective correlation functions at vanishing relative momenta of all particles. To facilitate the extraction of this limit from experimental data we derived in Eqs. (34) and (36) the most general parametrizations for the two- and three-pion correlation functions at small relative momenta. These new parametrizations are based on our expressions of the correlation functions in terms of the Wigner density of the source; they are exact up to second order in the relative momenta, i.e. for emission functions

$S(x, K)$ with a Gaussian x -dependence. After eliminating the redundant q -components, they are seen to depend on 16 parameters which are all functions of the average momentum \mathbf{K} of the pion pair resp. triplet. To determine all these parameter functions, a complete study of the two- and three-particle spectra as functions of all $6 + 9 = 15$ momentum components is necessary. (The 16th parameter, $\epsilon(K)$, describes the degree of coherence and enters the normalization of the correlation functions at vanishing relative momenta.) This is certainly not an easy task, and it might be worthwhile to study whether, for certain simple but not too unrealistic models for the emission function, it is not possible to obtain simpler parametrizations (for example by exploiting certain symmetries of the source).

Our results show that in the case of partially coherent sources the three-pion correlator contains independent information on the second space-time moments of the source which cannot be extracted from the two-pion correlator. This information is needed to separate the space-time characteristics (lengths of homogeneity or effective widths) of the chaotic and coherent parts of the emission function. To extract it in practice will not be easy, but the theoretical framework by which this should be done has been presented here.

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